Chapter 10

Recursion

Video Lecture

10.1 Recursion Fundamentals

Definition 10.1.1. Recursion is the process of finding smaller values and using them to calculate larger values.

Example

Jeff is climbing a 6 stair staircase. If he can climb 1 or 2 stairs at a time, in how many distinct ways can he get to the top?

Solution 10.1

We could do casework, but that would take a long time. Instead, let's solve this problem for small values first.

How many ways are there for Jeff to reach the 0th stair (ground)?

Jeff is already there at the start, so there is just 1 way to do that.

How many ways are there for Jeff to reach the 1st stair?

There is only 1 way because he must climb 1 step to reach there from the ground (0th stair).

How many ways are there for Jeff to reach the 2nd stair?

There are 2 ways: 2 single steps or 1 double step from ground (0th stair)

How can Jeff reach the 3rd stair?

Jeff either has to take a single step from the 2nd stair, or a double step from the 1st stair. Therefore, the number of ways is simply the number of ways to reach the first step plus the number of ways to reach the 2nd step, which is $1+2=3$.

How can use the same logic to find the number of ways to reach the nth stair?

To reach the nth stair, Jeff must take a 1 step from the $(n-1)$th stair or a double step from the $(n-2)$th stair. Therefore, the number of ways of reaching the nth step is the number of ways of reaching the $(n-1)$th plus the number of ways of reaching the $(n-2)$th stair.

Let's define a function, f(n), such that f(n) represents the number of ways to reach the nth step. Then, we can write the recursion

$$f(n) = f(n-1) + f(n-2)$$

.
The next step is to just iteratively calculate the values of $f(n)$ for all values of n until we find $f(6)$, since the staircase has 6 steps. We already calculated $f(1)$ and $f(2)$ so we can use them as base cases.

$f(1) = 1$
$f(2) = 2$
$f(3) = f(3-1) + f(3-2) = f(2) + f(1) = 2 + 1 = 3$
$f(4) = f(4-1) + f(4-2) = f(3) + f(2) = 3 + 2 = 5$
$f(5) = f(5-1) + f(5-2) = f(4) + f(3) = 5 + 3 = 8$
$f(6) = f(6-1) + f(6-2) = f(5) + f(4) = 8 + 5 = \boxed{13}$

Remark 10.1.2
Notice how the numbers follow the Fibonacci sequence. This is because the recursions for our sequence and the Fibonacci sequence are the same. However, this is not always the case for all recursion problems.

10.2 Recursion with Constraints

Example

Jeff is climbing a 7 stair staircase. Jeff can climb 1 or 2 stairs at a time on every step, but on the last step, he is allowed to climb 3 stairs. In how many distinct ways can he get to the top?

Video Solution
Answer:

Concept 10.2.1
To solve a problem with recursion requires a multi-step process:

1. Solve the problem for a few smaller base cases manually
2. In general, you will need as many base cases as the number of terms on the right hand side of your recursive function
3. For a given $f(n)$, figure out a recursive equation in terms of previous values of the

function by considering different ways to get to that point

4. Iteratively calculate values of $f(n)$ until you reach the desired number

Remark 10.2.2
You can also use f(0) = 1 as a base case because to get to the 0th stair, you have to take no steps and there is only 1 way to do so. This can be a little faster, however it could be a little confusing, so it won't be used in the examples. Nevertheless, feel free to use it in the problems section.

Example

Mike is climbing a staircase with 10 stairs. He is in a rush, so he will climb 2 or 3 steps at a time for most of the journey. If he reaches the 9th stair, he is allowed to climb 1 stair to reach the 10th stair. How many ways are there for Mike to reach the top of the staircase?

Solution 10.2

We can do a similar approach to the previous problem. We first begin by finding the base cases.
There is no way to reach the 1st stair since he cannot take a 1 step, so $f(1) = 0$.
There is 1 way to reach the 2nd stair by taking a double step, so $f(2) = 1$.
There is 1 way to reach the 3rd stair by taking a triple step. If the first step is a double step, then it's impossible to reach the 3rd stair since there are no single steps. Therefore, $f(3) = 1$.

Next, what is the recursive function for this problem?

To get the the nth stair, we must either take a 2 step from the $(n-2)$th stair or a 3 step from the $(n-3)$th stair. So, the recursion is

$$f(n) = f(n-2) + f(n-3)$$

Let's now evaluate the values of f(n)

$f(1) = 0$
$f(2) = 1$
$f(3) = 1$
$f(4) = f(4-2) + f(4-3) = f(2) + f(1) = 1$
$f(5) = f(5-2) + f(5-3) = f(3) + f(2) = 2$
$f(6) = f(6-2) + f(6-3) = f(4) + f(3) = 2$
$f(7) = f(7-2) + f(7-3) = f(5) + f(4) = 3$

$f(8) = f(8-2) + f(8-3) = f(6) + f(5) = 4$
$f(9) = f(9-2) + f(9-3) = f(7) + f(6) = 5$
$f(10) = f(10-2) + f(10-3) = f(8) + f(7) = 7$

So, is our answer just $f(10) = 7$?

No, because once he reaches stair 9, he is allowed to take a single step. So, we must add the number of ways to reach stair 9 to the number of ways to reach stair 8 and the number of ways to reach stair 7. Therefore, we must find:
$f(10) = f(8) + f(7) + f(9) = \boxed{12}$

Remark 10.2.3
In general, it never hurts to find too many base cases. In your recursion, if you run into any values you are not sure how to calculate (like f(2), f(3)), you can always go back and evaluate more base cases.

10.3 Probability Recursions

Example (AMC 8)

A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops what is the probability that the cricket has returned to the leaf where it started?

Video Solution
Answer:

Concept 10.3.1
For probability recursions, it can be a good idea to simply make a table and find the probability of being at each location every time interval. Remember to use the fact that the total probability at each hop must always be 1.

10.4 Practice Problems

Problem 10.4.1

Jeff is climbing a 7 step staircase. If he can climb 1 or 2 steps at a time, in how many ways can he get to the top?

Video Solution
Answer:

Problem 10.4.2 (AMC 8)

Everyday at school, Jo climbs a flight of 6 stairs. Jo can take the stairs 1, 2, or 3 at a time. For example, Jo could climb 3, then 1, then 2. In how many ways can Jo climb the stairs?

Video Solution
Answer:

Problem 10.4.3 (Omega Learn Math Contest)

Brandon is trying to reach the top of a 7 step staircase. For each jump except the last jump, he can either jump 1 or 3 steps forward. On the last jump, he can jump 1, 2, or 3 steps forward. How many different ways can he reach the top of the staircase?

Video Solution
Answer:

Problem 10.4.4 (AMC 12)

Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of 1, 2, 3, . . . , 12, including the empty set, are spacy?

Video Solution
Answer:

Problem 10.4.5 (AIME)

A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the following rules:

Any cube may be the bottom cube in the tower. The cube immediately on top of a cube with edge-length k must have edge-length at most k + 2. How many towers can be constructed?

Video Solution
Answer:

Additional Problems

Problem 10.4.6

Jeff is climbing a 10 step staircase. However, there are spiders on the 3rd and 9th steps, and he refuses to step on those stairs. If he can climb 1 or 2 steps at a time, in how many different ways can he get to the top?